PLUS ULTRA

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ABSTRACT. We define a reasonably well-behaved class of ultraimaginaries, i.e. classes modulo $\emptyset$-invariant equivalence relations, called *tame*, and establish some basic simplicity-theoretic facts. We also show feeble elimination of supersimple ultraimaginaries: If e is an ultraimaginary definable over a tuple a with $SU(a) < \omega^{\alpha+1}$, then e is eliminable up to rank $< \omega^\alpha$. Finally, we prove some uniform versions of the weak canonical base property.

1. INTRODUCTION

This paper arose out of an attempt to understand and generalize Chatzidakis' results on the weak canonical base property [6, Proposition 1.14 and Lemma 1.15]. In doing so, we realized that certain stability-theoretic phenomena were best explained using ultraimaginaries. It should be noted that ultraimaginaries occur naturally in simplicity theory and were in fact briefly considered in [3] before specializing to the more restricted class of almost hyperimaginaries. However, they have faded into oblivion since Ben Yaacov [1] has shown that no satisfactory independence theory can exist for them, as there are problems with both the finite character and the extension axiom for independence. Nevertheless, at least finite character can be salvaged if one restricts to quasi-finitary ultraimaginaries in a supersimple theory, or more generally to what we call *tame* ultraimaginaries.

We shall define ultraimaginaries in Section 2 and give various examples. We also give a first example of a natural general result involving them, Proposition 2.12, which for a supersimple theory of finite rank specializes to a theorem of Lascar. In Section 3 we define tame ultraimaginaries and recover certain tools from simplicity theory, even though, due to the lack of extension, canonical bases are not available

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in our context. One may thus hope to extend the techniques of this section for instance to the superrosy context, where the lack of canonical bases has been one of the main technical problems.

In Section 4 we prove feeble elimination of ultraimaginaries. In particular ultraimaginaries of finite rank are interbounded with hyperimaginaries. This is used in Section 5 to generalize some of Chatzidakis' results [6] on the weak canonical base property from sets of finite SU-rank to arbitrary ordinal SU-rank. It is interesting to compare this generalization to the coarser [10, Theorem 5.4] which uses α-closure. We expect that this is a general phenomenon: The use of ultraimaginaries allows for a more direct and more refined proof without explicit use of SU-rank; rank considerations principally intervene via the feeble elimination result of Section 4 and the technical results of Section 3.

All elements, tuples and parameter sets are hyperimaginary, unless stated otherwise. For an introduction to simplicity and hyperimaginaries, the reader is invited to consult [5], [8] or [13].

2. Ultraimaginaries

Definition 2.1. An *ultraimaginary* is the class a_E of a tuple a under an $\emptyset$-invariant equivalence relation E.

Note that tuples of ultraimaginaires are again ultraimaginary.

Definition 2.2. An ultraimaginary a_E is *definable* over an ultraimaginary b_F if any automorphism of the monster model stabilising the F-class of b also stabilises the E-class of a. It is *bounded* over b_F if the orbit of a under the group of automorphisms of the monster model which stabilise the F-class of b is contained in boundedly many E-classes. A *representative* of an ultraimaginary e is any hyperimaginary a such that e is definable over a. An ultraimaginary is *finitary* if it has a finite real representative. Two (tuples of) ultraimaginaries are *equivalent* over some set A of parameters if they are interdefinable over A.

Note that in contrast to hyperimaginaries, the class of a tuple of size κ modulo an $\emptyset$-invariant equivalence relation need not be equivalent to a tuple of ultraimaginaries with representatives of smaller size: Consider the equivalence relation on sequences of length κ of being equal except for a subsequence of smaller length.

Remark 2.3. As usual, if $E_A(x,y)$ is an A-invariant equivalence relation, one considers the $\emptyset$-invariant relation $E(xX,yY)$ given by

$$(X=Y\wedge X\equiv A\wedge E_X(x,y))\vee(X=Y\wedge x=y).$$

This is an equivalence relation, and $(aA)_E$ is equivalent to a_{E_A} over A.

Remark 2.4. As any $\emptyset$-invariant relation, E is given by a union of types over $\emptyset$.

Definition 2.5. We shall say that two ultraimaginaries *have the same (Lascar strong) type* over some set A if they have representatives which do. If the ambient theory is simple, we call two ultraimaginaries *independent* over A if they have representatives which are.

Clearly, two ultraimaginaries are conjugate by a (Lascar strong) automorphism over A if and only if they have the same (Lascar strong) type over A.

Remark 2.6. If e or e' is a sequence of ultraimaginaries, for $e \mathop{\smile\!\!\!\!\!\mid}\, e'$ to hold we require sequences of representatives which are independent. In particular, it is not clear even for real e' that an infinite sequence e of ultraimaginaries is independent of e' if every finite subsequence is independent of e'.

Ultraimaginaries arise quite naturally in stability and simplicity theory.

Example 2.7. Let $p_A\in S(A)$ be a regular type in a stable theory. For $A',A''\models \mathrm{tp}(A)$ put $E(A',A'')$ if $p_{A'}\not\perp p_{A''}$. Then E is an $\emptyset$-invariant equivalence relation, and A_E codes the non-orthogonality class of p_A.

The work with ultraimaginaries requires caution, as some basic properties become problematic.

Example 2.8. [1] Let E be the $\emptyset$-invariant equivalence relation on infinite sequences which holds if they differ only on finitely many elements. Consider a sequence $I=(a_i:i<\omega)$ of elements such that no finite subtuple is bounded over the remaining elements. Then every finite tuple $\bar a\in I$ can be moved to a disjoint conjugate over I_E, but I cannot. Similarly, if I is a Morley sequence in a simple theory, then $\bar a \mathop{\smile\!\!\!\!\!\mid}\, I_E$ for any finite $\bar a\in I$, but $I \not\mathop{\smile\!\!\!\!\!\mid}\, I_E$.

Even in the ω-stable context, for classes of finite tuples, the theory is not smooth.

Example 2.9. Let T be the theory of a cycle-free graph (forest) of infinite valency, with predicates $P_n(x, y)$ for couples of points of distance n for all $< \omega$. It is easy to see by back-and-forth that T eliminates quantifiers and is ω-stable of rank ω; the formula $P_n(x, a)$ has rank n over a. Let E be the $\emptyset$-invariant equivalence relation of being in the same connected component. Then existence of non-forking extensions fails over a_E, as any two points in the connected component of a have some finite distance n, and hence rank n over one another, but rank $\geq k$ over a_E for all $k < \omega$, since a_E is definable over any point of distance at least k.

The same phenomenon can be observed for any type p of rank $SU(p) = \omega$ in a simple theory, with the relation $E(x, y)$ on p which holds if $SU(x/y) < \omega$ and $SU(y/x) < \omega$ (actually, one follows from the other by the Lascar inequalities).

The behaviour of Example 2.9 is inconvenient and signifies that we shall avoid considering types *over* an ultraimaginary. The behaviour of Example 2.8 is outright vexatious; we shall restrict the class of ultraimaginaries under consideration in order to preserve the finite character of independence.

Definition 2.10. An ultraimaginary e is *quasi-finitary* if there is a finite real tuple a such that e is bounded over a.

For hyperimaginary tuples contained in the bounded closure of a finite set, we shall use *quasi-finite* rather than *quasi-finitary*, in order to emphasize the distinction between usual hyperimaginaries and ultraimaginaries. The set of all (quasi-finitary) ultraimaginaries definable over some ultraimaginary set E will be denoted by $\mathrm{dcl}^u(E)$ (or $\mathrm{dcl}^{qfu}(E)$, respectively). Similarly, $\mathrm{bdd}^u(E)$ and $\mathrm{bdd}^{qfu}(E)$ will denote the corresponding bounded closures. If A is a set of representatives for E, then the number of ultraimaginaries with representatives of length κ in $\mathrm{bdd}^u(E)$ (and *a fortiori* in the other closures as well) is bounded in terms of the number of Lascar strong types over A of real tuples of length κ, since equality of Lascar strong type over A is the finest bounded A-invariant equivalence relation.

Remark 2.11. If e is a quasi-finitary ultraimaginary then $\mathrm{bdd}^u(e) = \mathrm{bdd}^{qfu}(e)$ and $\mathrm{dcl}^u(e) = \mathrm{dcl}^{qfu}(e)$.

Proposition 2.12. *The following are equivalent for two ultraimaginaries a and b:*

(1) $\mathrm{bdd}^u(a) \cap \mathrm{bdd}^u(b) = \mathrm{bdd}^u(\emptyset)$.

(2) *For any* $a' \equiv^{lstp} a$ *there is* $n < \omega$ *and a sequence* $(a_i b_i : i \leq n)$ *such that*
$$a_0 = a, \quad b_0 = b, \quad a_n = a'$$
and for each $i < n$
$$\mathrm{bdd}^u(a_i) b_{i+1} \equiv \mathrm{bdd}^u(a_i) b_i \quad \textit{and} \quad a_{i+1} \mathrm{bdd}^u(b_{i+1}) \equiv a_i \mathrm{bdd}^u(b_{i+1}).$$

If a or b is quasi-finite, this is also eqivalent to $\mathrm{bdd}^{qfu}(a) \cap \mathrm{bdd}^{qfu}(b) = \mathrm{bdd}^{qfu}(\emptyset)$.

Proof: (1) $\Rightarrow$ (2) Suppose $\mathrm{bdd}^u(a) \cap \mathrm{bdd}^u(b) = \mathrm{bdd}^u(\emptyset)$, and define an $\emptyset$-invariant equivalence relation on $\mathrm{lstp}(ab)$ by $E(xy, x'y')$ if there is a sequence $(x_i y_i : i \leq n)$ such that $x_0 y_0 = xy$, $x_n y_n = x'y'$, and for each $i < n$ we have
$$\mathrm{bdd}^u(x_i) y_{i+1} \equiv \mathrm{bdd}^u(x_i) y_i \quad \text{and} \quad x_{i+1} \mathrm{bdd}^u(y_{i+1}) \equiv x_i \mathrm{bdd}^u(y_{i+1}).$$
Now if $\mathrm{bdd}^u(a) b' \equiv \mathrm{bdd}^u(a) b$, then $\models E(ab, ab')$. Hence $(ab)_E \in \mathrm{bdd}^u(a)$. Similarly $(ab)_E \in \mathrm{bdd}^u(b)$, whence $(ab)_E \in \mathrm{bdd}^u(\emptyset)$. But for any $a' \equiv^{lstp} a$ there is b' with $ab \equiv^{lstp} a'b'$. Then $\models E(ab, a'b')$, in particular (2) holds.

(2) $\Rightarrow$ (1) Suppose not, and consider $e \in (\mathrm{bdd}^u(a) \cap \mathrm{bdd}^u(b)) \setminus \mathrm{bdd}^u(\emptyset)$. As $e \notin \mathrm{bdd}^u(\emptyset)$ there is $a' \models \mathrm{lstp}(a)$ with $e \notin \mathrm{bdd}^u(a')$. Consider a sequence $(a_i, b_i : i \leq n)$ as in (2). Since $\mathrm{bdd}^u(a_i) b_{i+1} \equiv \mathrm{bdd}^u(a_i) b_i$ and $a_{i+1} \mathrm{bdd}^u(b_{i+1}) \equiv a_i \mathrm{bdd}^u(b_{i+1})$ we have
$$\begin{aligned} \mathrm{bdd}^u(a_i) \cap \mathrm{bdd}^u(b_i) &= \mathrm{bdd}^u(a_i) \cap \mathrm{bdd}^u(b_{i+1}) \\ &= \mathrm{bdd}^u(a_{i+1}) \cap \mathrm{bdd}^u(b_{i+1}). \end{aligned}$$
In particular,
$$\begin{aligned} e \in \mathrm{bdd}^u(a) \cap \mathrm{bdd}^u(b) &= \mathrm{bdd}^u(a_0) \cap \mathrm{bdd}^u(b_0) \\ &= \mathrm{bdd}^u(a_n) \cap \mathrm{bdd}^u(b_n) \subseteq \mathrm{bdd}^u(a'), \end{aligned}$$
a contradiction.

The last assertion follows from Remark 2.11. $\square$

Remark 2.13. For hyperimaginary a and ultraimaginary b and c the condition $\mathrm{bdd}^u(a) b \equiv \mathrm{bdd}^u(a) c$ is equivalent to $b \equiv_a^{lstp} c$.

Using weak elimination of ultraimaginaries proven in Section 4, we recover a Lemma of Lascar [7] (see also [9, Lemma 2.2]), proved originally for stable theories of finite Lascar rank.

Corollary 2.14. *Let T be a simple theory of finite SU-rank and a, b finite imaginary tuples. The following are equivalent:*

(1) $\mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(b) = \mathrm{acl}^{eq}(\emptyset)$.
(2) *For any* $a' \equiv^{lstp} a$ *independent of* a *there are sequences* $a = a_0, \ldots, a_n = a'$ *and* $b = b_0, \ldots, b_n$, *such that* $b_{i+1} \equiv^{lstp}_{a_i} b_i$ *and* $a_{i+1} \equiv^{lstp}_{b_{i+1}} a_i$ *for each* $i < n$.

Proof: By Theorem 4.6 supersimple theories of finite rank have weak elimination of quasi-finitary ultraimaginaries; by [4] supersimple theories eliminate hyperimaginaries. Hence condition (1) is equivalent to $\mathrm{bdd}^u(a) \cap \mathrm{bdd}^u(b) = \mathrm{bdd}^u(\emptyset)$. By Remark 2.13 condition (2) is equivalent to condition (2) of Proposition 2.12. So (1) $\Rightarrow$ (2) follows from Proposition 2.12; for the converse given arbitrary $a' \equiv^{lstp}_A a$ we consider $a'' \equiv^{lstp}_A a$ with $a'' \mathop{\smile\!\!\!\!\!\mid}_A aa'$ and compose the sequence $(a_ib_i : i \leq n)$ from $ab = a_0b_0$ to $a_n = a''$ with the sequence $(a_ib_i : n \leq i \leq \ell)$ from a_nb_n to $a_\ell = a'$. Hence (2) holds for arbitrary $a' \equiv^{lstp}_A a$, so we can again apply Proposition 2.12. $\square$

3. Ultraimaginaries in simple theories

From now on the ambient theory will be simple. Our notation is standard and follows [13]. We shall be working in a sufficiently saturated model of the ambient theory. Tuples are again tuples of hyperimaginaries, and closures (definable and bounded closures) will include hyperimaginaries.

Remark 3.1. Since in a simple theory Lascar strong type equals Kim-Pillay strong type, we have $\mathrm{bdd}^u(A) = \mathrm{dcl}^u(\mathrm{bdd}(A))$. But of course, as with real and imaginary algebraic closures, $\mathrm{bdd}(A) \cap \mathrm{bdd}(B) = \mathrm{bdd}(\emptyset)$ does not imply $\mathrm{bdd}^u(A) \cap \mathrm{bdd}^u(B) = \mathrm{bdd}^u(\emptyset)$ unless the theory weakly eliminates ultraimaginaries.

In a simple theory, ultraimaginary independence is clearly symmetric, and satisfies local character and extension (but recall that we only consider hyperimaginary base sets), since this is inherited from suitable representatives. As for transitivity, we have the following.

Fact 3.2. [3, Lemma 1.10] *Let* A, a *be hyperimaginary, and* e, e', e'' *ultraimaginary.*

- *If* $e \mathop{\smile\!\!\!\!\!\mid}_A e'e''$ *and* $e' \mathop{\smile\!\!\!\!\!\mid}_A e''$, *then* $ee' \mathop{\smile\!\!\!\!\!\mid}_A e''$ *and* $e \mathop{\smile\!\!\!\!\!\mid}_A e'$.
- $e \mathop{\smile\!\!\!\!\!\mid}_A ae'$ *if and only if* $e \mathop{\smile\!\!\!\!\!\mid}_A a$ *and* $e \mathop{\smile\!\!\!\!\!\mid}_{Aa} e'$.

The Independence Theorem and Boundedness axiom also hold.

Fact 3.3. [3, page 189] *Let A be hyperimaginary and e, e' ultraimaginary with $e \mathop{\smile\!\!\!\!\!\!|}_A e'$.*

- *If f, f' are ultraimaginary with $f \mathop{\smile\!\!\!\!\!\!|}_A e$, $f' \mathop{\smile\!\!\!\!\!\!|}_A e'$ and $f \equiv^{Lstp}_A f'$, then there is $f'' \mathop{\smile\!\!\!\!\!\!|}_A ee'$ with $ef'' \equiv^{Lstp}_A ef$ and $e'f'' \equiv^{Lstp}_A e'f'$.*
- *If $e'' \in \mathrm{bdd}^u(Ae)$ then $e'' \mathop{\smile\!\!\!\!\!\!|}_A e'$. Moreover, if $e \mathop{\smile\!\!\!\!\!\!|}_a e$ for every representative a of an ultraimaginary e'', then $e \in \mathrm{bdd}^u(e'')$.*

Next, ultraimaginary bounded closures of independent sets intersect trivially.

Lemma 3.4. *If A is hyperimaginary and e, e' ultraimaginary with $e \mathop{\smile\!\!\!\!\!\!|}_A e'$, then $\mathrm{bdd}^u(Ae) \cap \mathrm{bdd}^u(Ae') = \mathrm{bdd}^u(A)$.*

Proof: Replacing e and e' by A-independent representatives, we may assume that e and e' are hyperimaginary. Consider $a_E \in \mathrm{bdd}^u(Ae) \cap \mathrm{bdd}^u(Ae')$. We may assume $a \mathop{\smile\!\!\!\!\!\!|}_{Ae} e'$, whence $ae \mathop{\smile\!\!\!\!\!\!|}_A e'$. Let $(a_i : i < \omega)$ be a Morley sequence in $\mathrm{lstp}(a/Ae')$. Then $E(a_i, a_j)$ for all $i, j < \omega$. But $a_i \mathop{\smile\!\!\!\!\!\!|}_A a_j$ for $i \neq j$, so $\pi(x, a_j) = \mathrm{tp}(a_i/a_j)$ does not fork over A, and neither does $\pi(x, a)$. Note that $\pi(x, y)$ implies $E(x, y)$.

Now suppose $a_E \notin \mathrm{bdd}^u(A)$. We can then find a long sequence $(a'_i : i < \alpha)$ of A-conjugates of a such that $\neg E(a'_i, a'_j)$ for $i \neq j$. By the Erdős-Rado theorem and compactness (see e.g. [5, Proposition 1.6]) there is an infinite A-indiscernible sequence $(a''_i : i < \omega)$ whose 2-type over A is among the 2-types of $(a'_i : i < \alpha)$. In particular $\neg E(a''_i, a''_j)$ for $i \neq j$, and $(\pi(x, a''_i) : i < \omega)$ is 2-inconsistent. Since $a''_0 \models \mathrm{tp}(a/A)$, we see that $\pi(x, a)$ divides over A, a contradiction. □

As we have seen in Remark 2.6, finite character may fail for ultraimaginaries. The next definition singles out the subclass of ultraimaginaries where this does not happen, at least for hyperimaginary sets.

Definition 3.5. Let T be simple. An ultraimaginary e is *tame* if for all sets A, B of hyperimaginaries we have $e \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $e \mathop{\smile\!\!\!\!\!\!|}_A B_0$ for all finite subsets $B_0 \subseteq B$. It is *supersimple* if it has a representative of ordinal SU-rank.

Remark 3.6. A supersimple ultraimaginary in a simple theory is quasi-finitary; in a supersimple theory the converse holds as well.

Proof: Suppose A is a representative for an ultraimaginary e with $SU(A) < \infty$, and let B be a real tuple with $A \in \mathrm{bdd}(B)$. Let $b \in B$ be a finite subtuple with $SU(A/b)$ minimal; it follows that $A \mathop{\smile\!\!\!\!\!\!|}_b B$.

Hence $A \subseteq \operatorname{bdd}(b)$ and e is bounded over b, so e is quasi-finitary. In a supersimple theory the converse is obvious. □

We are really interested in the set of tame ultraimaginaries. However, we do not have a good criterion when an ultraimaginary is tame; moreover, an ultraimaginary definable over a tame ultraimaginary need not be tame itself. For instance, the sequence I in Example 2.8 is tame (since it is real), but I_E is not. Clearly, an ultraimaginary definable (or even bounded) over a quasi-finitary/supersimple ultraimaginary is itself quasi-finitary/supersimple.

Lemma 3.7. *A supersimple ultraimaginary is tame. In particular, quasi-finitary ultraimaginaries in a supersimple theory are tame.*

Proof: Let e be a supersimple ultraimaginary, and a a representative with $SU(a) < \infty$. Consider sets A and B. There is a finite $b \in B$ with $a \mathop{\smile\!\!\!\!\!\!\,|}_{Ab} B$. So $e \mathop{\smile\!\!\!\!\!\!\,|}_A B$ if and only if $e \mathop{\smile\!\!\!\!\!\!\,|}_A b$ by Fact 3.2. Thus e is tame. □

In a supersimple theory quasi-finitary ultraimaginaries are the correct ones to consider: Due to elimination of hyperimaginaries all parameters consist of imaginaries of ordinal SU-rank; as canonical bases of such imaginaries are finite, we can always reduce to a quasi-finitary situation.

Another kind of tame ultraimaginaries arose in the generalization of the group configuration theorem to simple theories [2, 3].

Definition 3.8. An invariant equivalence relation E is *almost type-definable* if there is a type-definable symmetric and reflexive relation R finer than E such that any E-class can be covered by boundedly many R-balls (i.e. sets of the form $\{x : xRa\}$ for varying a). A class modulo an almost type-definable equivalence relation is called an *almost hyperimaginary.*

Fact 3.9. [3, page 188] *Almost hyperimaginaries are tame. In fact, they satisfy finite character.*

We shall now consider how to obtain invariant equivalence relations, and hence ultraimaginaries.

Proposition 3.10. *Let T be stable. For algebraically closed A and an $\emptyset$-invariant equivalence relation E on* $\operatorname{tp}(b)$, *consider the relation* $R(X, Y)$ *given by*

$$\exists xy\,[Xx \equiv Yy \equiv Ab \wedge x \mathop{\smile\!\!\!\!\!\!\,|}_X Y \wedge y \mathop{\smile\!\!\!\!\!\!\,|}_Y X \wedge E(x, y)].$$

Then R is an $\emptyset$-invariant equivalence relation on $\mathrm{tp}(A)$.

Proof: Clearly, R is $\emptyset$-invariant, reflexive and symmetric. So suppose that $R(A,A')$ and $R(A',A'')$ both hold, and let this be witnessed by b,b' and b^*,b''. Let $b_1 \models \mathrm{tp}(b'/A') = \mathrm{tp}(b^*/A')$ with $b_1 \mathop{\smile\!\!\!\!\!\!\mid}_{A'} AA''$. Since A' is algebraically closed, $b' \mathop{\smile\!\!\!\!\!\!\mid}_{A'} A$ and $b^* \mathop{\smile\!\!\!\!\!\!\mid}_{A'} A''$ we have $b_1 \equiv_{AA'} b'$ and $b_1 \equiv_{A'A''} b^*$ by stationarity. Hence there are b_0, b_2 with $bb' \equiv_{AA'} b_0b_1$ and $b^*b'' \equiv_{A'A''} b_1b_2$. In particular $E(b_0,b_1)$ and $E(b_1,b_2)$ hold, and so does $E(b_0,b_2)$. Moreover, we may assume $b_0 \mathop{\smile\!\!\!\!\!\!\mid}_{AA'b_1} A''$ and $b_2 \mathop{\smile\!\!\!\!\!\!\mid}_{A'A''b_1} A'$. Now $b_1 \mathop{\smile\!\!\!\!\!\!\mid}_{A'} AA''$ implies $b_0 \mathop{\smile\!\!\!\!\!\!\mid}_{AA'} A''$ and $b_2 \mathop{\smile\!\!\!\!\!\!\mid}_{A'A''} A$. Then $b_0 \mathop{\smile\!\!\!\!\!\!\mid}_{A} A'$ and $b_2 \mathop{\smile\!\!\!\!\!\!\mid}_{A''} A'$ imply $b_0 \mathop{\smile\!\!\!\!\!\!\mid}_{A} A''$ and $b_2 \mathop{\smile\!\!\!\!\!\!\mid}_{A''} A$, whence $R(A,A'')$ holds. So R is transitive. □

Remark 3.11. Recall that a reflexive and symmetric binary relation $R(x,y)$ on a partial type $\pi(x)$ is *generically transitive* if whenever $x,y,z \models \pi$ and $x \mathop{\smile\!\!\!\!\!\!\mid}_{y} z$, then $R(x,y)$ and $R(y,z)$ together imply $R(x,z)$. If T is merely simple, the relation R in Proposition 3.10 is still generically transitive. However, contrary to the type-definable case [13, Lemma 3.3.1], the two-step iterate of an $\emptyset$-invariant, reflexive, symmetric and generically transitive relation on a Lascar strong type need not be transitive.

Example 3.12. Consider on the forest of Example 2.9 the relation $R(a,b)$ which holds if 3 divides the distance between a and b. This is generically transitive, as for $a' \mathop{\smile\!\!\!\!\!\!\mid}_{a} a''$ the distance between a' and a'' is the sum of the distances between a' and a and between a and a''. However, two points of distance 2 are easily seen to be R^2-related, so the transitive closure E of R is just the relation of being in the same connected component. But no two points of distance 1 are R^2-related.

Clearly, in the above example the three-step iterate R^3 is transitive, as it is just the relation of being connected. Is there an example of a generically transitive symmetric and reflexive $\emptyset$-invariant relation R such that R^n is not transitive for any $n<\omega$?

The next proposition shows that in a simple theory and under some conditions, if R is a generically transitive reflexive and symmetric relation, then at least its transitive closure is not bounded, unless R holds for two independent elements. We first recall the definitions of SU_p-rank and p-closure.

Definition 3.13. [13, Remark 5.1.19] Let P be an $\emptyset$-invariant family of regular types closed under nonforking extensions. The *SU_P-rank* is the smallest function from the collection of all types to the ordinals

together with infinity, such that $SU(a/A) \geq \alpha + 1$ if there is some $B \supseteq A$ and some $c \not\downfree_B a$ with $\mathrm{tp}(c/B) \in P$.
The *P-closure* of a set A is given by $\mathrm{cl}_P(A) = \{a : SU_P(a/A) = 0\}$.

Then SU_P-rank satisfies the Lascar inequalities [13, Exercise 5.1.20]. Note that unless P contains all non-orthogonality classes of types of SU-rank 1, the P-closure has the size of the monster model.

Fact 3.14. [13, Lemma 3.5.3] *The following are equivalent:*

1. $\mathrm{tp}(a/A)$ *is foreign to all types q with $SU_P(q) = 0$.*
2. $a \downfree_A \mathrm{cl}_P(A)$.
3. $a \downfree_A \mathrm{dcl}(aA) \cap \mathrm{cl}_P(A)$.
4. $\mathrm{dcl}(aA) \cap \mathrm{cl}_P(A) \subseteq \mathrm{bdd}(A)$.

Note that P-closure is well-behaved with respect to independence:

Fact 3.15. [13, Lemma 3.5.5] and [14, Lemma 3] *Suppose $A \downfree_B C$. Then $\mathrm{cl}_P(A) \downfree_{\mathrm{cl}_P(B)} \mathrm{cl}_P(C)$. More precisely, for any $A_0 \subseteq \mathrm{cl}_P(A)$ we have $A_0 \downfree_{B_0} \mathrm{cl}_P(C)$, where $B_0 = \mathrm{dcl}(A_0B) \cap \mathrm{cl}_P(B)$. In particular, $\mathrm{cl}_P(AB) \cap \mathrm{cl}_P(BC) = \mathrm{cl}_P(B)$.*

Remark 3.16. If $p \in S(\emptyset)$ is regular and P is the family of all nonforking extensions of p, we shall write SU_p and cl_p. Another choice for P is the family of all regular types of SU-rank ω^α used in the proof of Theorem 4.6. One can also take P to be the family of types foreign to some $\emptyset$-invariant collection Σ of partial types; in this case P-closure cl_P is equal to the Σ-closure cl_Σ defined in Definition 5.2 (see [14]).

Lemma 3.17. *Let P be an $\emptyset$-invariant family of regular types closed under nonforking extensions. Suppose $SU_P(a/bc) = SU_P(a/b) = n$ is finite. Then $\mathrm{cl}_P(a) \downfree_{\mathrm{cl}_P(b)} \mathrm{cl}_P(c)$.*

Proof: Let $(a_i : i \leq \omega)$ be a Morley sequence in $\mathrm{lstp}(a/bc)$ with $a = a_\omega$, and put $c_k = (a_i : i < n)$ and $d_k = \mathrm{Cb}(a/bc_k)$. We shall show first that $d_k \in \mathrm{cl}_P(b)$.

Let $(a'_i : i < \omega)$ be a Morley sequence in $\mathrm{lstp}(a/bc_k)$. Since

$$SU_P(d_k/b) \leq SU_P(c_k/b) \leq nk$$

is finite, there is $\ell < \omega$ and $r \geq 0$ with $SU_P(d_k/b, a'_i : i < \ell') = r$ for all $\ell' \geq \ell$. Suppose $r > 0$. Then there is $B \supseteq \{b, a'_i : i < \ell\}$ and b' with $\mathrm{tp}(b'/B) \in P$ such that $b' \not\downfree_B d_k$; we may assume

$$Bb' \downfree_{b,d_k,(a'_i:i<\ell)} (a'_i : i < \omega).$$

As $d_k \in \mathrm{dcl}(a'_i : i < \omega)$, there is minimal $\ell' \geq \ell$ such that $b' \not\mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_B (a'_i : i \leq \ell')$. So

$$\begin{aligned} SU_P(a'_i : i \leq \ell'/b, d_k, a'_i : i < \ell) &= SU_P(a'_i : i \leq \ell'/Bb') \\ &< SU_P(a_i : i \leq \ell'/B) \\ &\leq SU_P(a'_i : i \leq \ell'/b, a'_i : i < \ell) \\ &\leq \ell n. \end{aligned}$$

By Lascar symmetry (the second Lascar inequality),

$$r = SU_P(d_k/b, a'_i : i \leq \ell') < SU_P(d_k/b, a'_i : i < \ell) = r,$$

a contradiction. Thus $r = 0$. By the Lascar inequalites,

$$\begin{aligned} \ell n + SU_P(d_k/b) &= SU_P(a_i : i < \ell/b, d_k) + SU_P(d_k/b) \\ &\leq SU_P(d_k, a_i : i < \ell/b) \\ &\leq SU_P(d_k/b, a_i : i < \ell) \oplus SU_P(a_i : i < \ell/b) \\ &= 0 \oplus \ell n, \end{aligned}$$

whence $SU_P(d_k/b) = 0$ and $d_k \in \mathrm{cl}_P(b)$.

Since $a \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{bd_k} (a_i : i < k)$ Fact 3.15 yields $a \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{\mathrm{cl}_P(b)} (a_i : i < k)$ for all k, whence $a \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{\mathrm{cl}_P(b)} (a_i : i < \omega)$. If $d = \mathrm{Cb}(a/bc)$, then $d \in \mathrm{dcl}(a_i : i < \omega)$ and $a \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{bd} c$. So $\mathrm{cl}_P(a) \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{\mathrm{cl}_P(b)} \mathrm{cl}_P(bd)$ and $\mathrm{cl}_P(a) \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{\mathrm{cl}_P(bd)} \mathrm{cl}_P(c)$ by Fact 3.15; the result follows by transitivity. $\square$

Proposition 3.18. *Let T be simple. Suppose R is an $\emptyset$-invariant, reflexive, symmetric and generically transitive relation on* $\mathrm{lstp}(a)$*, and P is an $\emptyset$-invariant family of regular types closed under non-forking extensions such that $SU_P(a)$ is finite. Let E be the transitive closure of R, and suppose $a_E \in \mathrm{bdd}^u(\mathrm{cl}_P(\emptyset))$. Then there is $a' \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{\mathrm{cl}_P(\emptyset)} a$ with $R(a, a')$.*

Proof: Put $c = \mathrm{bdd}(a) \cap \mathrm{cl}_P(\emptyset)$. Then $a \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_c \mathrm{cl}_P(\emptyset)$ by Fact 3.14, whence $a_E \in \mathrm{bdd}^u(c)$ by Lemma 3.4. Let $a' \equiv^{lstp}_c a$ with $a' \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_c a$. Then $a_E = a'_E$, so there is $n < \omega$ and a chain $a = a_0, a_1, \ldots, a_n = a'$ such that $R(a_i, a_{i+1})$ holds for all $i < n$. Put $a'_0 = a_0$, and for $0 < i < n$ let

$$a'_i \equiv^{lstp}_{a_i} a'_{i-1} \quad \text{with} \quad a'_i \mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}_{a_i} a_{i+1}.$$

Claim. $\mathrm{bdd}^u(a'_i) \cap \mathrm{bdd}^u(a_{i+1}) \subseteq \mathrm{bdd}^u(a_0)$.

Proof of Claim: For $i = 0$ this is trivial. For $i > 0$, as $a'_i \equiv^{lstp}_{a_i} a'_{i-1}$ and $\mathrm{bdd}^u(a_i) = \mathrm{dcl}^u(\mathrm{bdd}(a_i))$, we get

$$\mathrm{bdd}^u(a'_i) \cap \mathrm{bdd}^u(a_i) = \mathrm{bdd}^u(a'_{i-1}) \cap \mathrm{bdd}^u(a_i).$$

Next, $a_i' \mathop{\smile\!\!\!\!\!\!\,|}_{a_i} a_{i+1}$ implies

$$\operatorname{bdd}^u(a_i'a_i) \cap \operatorname{bdd}^u(a_ia_{i+1}) = \operatorname{bdd}^u(a_i)$$

by Lemma 3.4. Hence inductively

$$\begin{aligned}\operatorname{bdd}^u(a_i') \cap \operatorname{bdd}^u(a_{i+1}) &\subseteq \operatorname{bdd}^u(a_i') \cap \operatorname{bdd}^u(a_i)\\ &= \operatorname{bdd}^u(a_{i-1}') \cap \operatorname{bdd}^u(a_i)\\ &\subseteq \operatorname{bdd}^u(a_0). \quad \square\end{aligned}$$

Now by generic transitivity and induction, $R(a_i', a_{i+1})$ holds for all $i < n$. In particular $R(a_{n-1}', a_n)$ holds, and by Lemma 3.4

$$\operatorname{bdd}^u(a_{n-1}') \cap \operatorname{bdd}^u(a_n) \subseteq \operatorname{bdd}^u(a_0) \cap \operatorname{bdd}^u(a_n) = \operatorname{bdd}^u(c).$$

Choose a'' with $R(a'', a_{n-1}')$ such that $SU_P(a''/a_{n-1}')$ is maximal possible. We may choose it such that $a'' \mathop{\smile\!\!\!\!\!\!\,|}_{a_{n-1}'} a_n$. Then

$$\operatorname{bdd}^u(a'') \cap \operatorname{bdd}^u(a_n) \subseteq \operatorname{bdd}^u(a_n) \cap \operatorname{bdd}^u(a_{n-1}') \subseteq \operatorname{bdd}^u(c)$$

and

$$SU_P(a''/a_n) \geq SU_P(a''/a_{n-1}'a_n) = SU_P(a''/a_{n-1}').$$

Rename $a''a_n$ as a_1a_2, and note that $\operatorname{bdd}^u(a_1) \cap \operatorname{bdd}(a_2) \subseteq \operatorname{bdd}^u(c)$, $c \subseteq \operatorname{bdd}(a_2)$, and $SU_P(a_1/a_2)$ is maximal possible among tuples (x, y) with $R(x, y)$. Moreover,

$$\begin{aligned}SU_P(a_2/a_1) &= SU_P(a_1a_2) - SU_P(a_1)\\ &= SU_P(a_1a_2) - SU_P(a_2) = SU_P(a_1/a_2),\end{aligned}$$

so this is also maximal.

Choose $a_3 \mathop{\smile\!\!\!\!\!\!\,|}_{a_2} a_1$ with $a_3 \equiv^{lstp}_{a_2} a_1$. By generic transitivity $R(a_1, a_3)$ holds. Moreover,

$$SU_P(a_3/a_1) \geq SU_P(a_3/a_1a_2) = SU_P(a_3/a_2),$$

so equality holds. Similarly,

$$SU_P(a_1/a_3) = SU_P(a_1/a_2a_3) = SU_P(a_1/a_2).$$

Now $SU_P(a_i/a_j) = SU_P(a_i/a_ja_k)$ for $\{i, j, k\} = \{1, 2, 3\}$ implies by Lemma 3.17 that

$$\operatorname{cl}_P(a_i) \mathop{\smile\!\!\!\!\!\!\,|}_{\operatorname{cl}_P(a_j)} \operatorname{cl}_P(a_k).$$

In particular,

$$\operatorname{cl}_P(a_i) \cap \operatorname{cl}_P(a_k) = \operatorname{cl}_P(a_1) \cap \operatorname{cl}_P(a_2) \cap \operatorname{cl}_P(a_3).$$

Let $b = \operatorname{cl}_P(a_1) \cap \operatorname{cl}_P(a_2) \cap \operatorname{bdd}(a_1a_2)$. Then $\operatorname{cl}_P(a_1) \cap \operatorname{cl}_P(a_2) = \operatorname{cl}_P(b)$ by [10, Lemma 3.18]. Let $F(x, y)$ be the $\emptyset$-invariant equivalence relation on $\mathrm{lstp}(b)$ given by $\mathrm{cl}_P(x) = \mathrm{cl}_P(y)$. As b_F is fixed by the $\mathrm{bdd}(a_2)$-automorphism moving a_1 to a_3 and $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_{a_2} a_3$, we get $b_F \in \mathrm{bdd}^u(a_2)$ by Lemma 3.4. Similarly, considering an $a_3' \mathop{\smile\!\!\!\!\!\!\mid}_{a_1} a_2$ with $a_3' \equiv^{lstp}_{a_1} a_2$ we obtain $b_F \in \mathrm{bdd}^u(a_1)$, whence

$$b_F \in \mathrm{bdd}^u(a_1) \cap \mathrm{bdd}^u(a_2) \subseteq \mathrm{bdd}^u(c).$$

So if $b' \models lstp(b/c)$ with $b' \mathop{\smile\!\!\!\!\!\!\mid}_c b$, then $b'_F = b_F$ and

$$\mathrm{cl}_P(b') = \mathrm{cl}_P(b) = \mathrm{cl}_P(c) = \mathrm{cl}_P(\emptyset).$$

But now

$$\mathrm{Cb}(a_3/\mathrm{cl}_P(a_1)\mathrm{cl}_P(a_2)) \subseteq \mathrm{cl}_P(a_1) \cap \mathrm{cl}_P(a_2) = \mathrm{cl}_P(b) = \mathrm{cl}_P(\emptyset),$$

so $a_3 \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_P(\emptyset)} a_2$, as required. $\square$

We shall illustrate the use of the proposition in Propositions 3.26 and 3.28, whose proof in the hyperimaginary case uses canonical bases. From now on, let Σ be an $\emptyset$-invariant family of partial types. Recall first the definitions of internality, analysability and orthogonality for hyperimaginaries.

Definition 3.19. Let π be a partial type over A. Then π is

- (*almost*) Σ-*internal* if for every realization a of π there is $B \mathop{\smile\!\!\!\!\!\!\mid}_A a$ and a tuple $\bar{b}$ of realizations of types in Σ based on B, such that $a \in \mathrm{dcl}(B\bar{b})$ (or $a \in \mathrm{bdd}(B\bar{b})$, respectively).
- Σ-*analysable* if for any realization a of π there is a sequence $(a_i : i < \alpha)$ such that $\mathrm{tp}(a_i/A, a_j : j < i)$ is Σ-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A, a_i : i < \alpha)$.

Finally, $p \in S(A)$ is *orthogonal* to $q \in S(B)$ if for all $C \supseteq AB$, $a \models p$, and $b \models q$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_A C$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_B C$ we have $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$. The type p is *orthogonal to* B if it is orthogonal to all types over B.

Note that in the definition of analysability, we may in addition require $a_i \in \mathrm{dcl}(Aa)$ for all $i < \alpha$. We now generalize these notions to ultraimaginaries.

Definition 3.20. We shall say that an ultraimaginary e is (*almost*) Σ-*internal*, or is Σ-*analysable*, if it has a representative which is. Similarly, e is *orthogonal* over A to some type p if for all $B \mathop{\smile\!\!\!\!\!\!\mid}_A e$ such that p is over B and for any realization $b \models p|B$ we have $e \mathop{\smile\!\!\!\!\!\!\mid}_A Bb$.

Remark 3.21. This definition does not imply that we define the notion of an analysis of an ultraimaginary. Moreover, e orthogonal to p over A need not imply that e has a representative which is orthogonal to p.

And unless e is tame, orthogonality of e over A to p need not imply orthogonality to $p^{(\omega)}$.

Definition 3.22. For an ordinal α the *α-th Σ-level* of a over A is defined inductively by $\ell_0^\Sigma(a/A) = \mathrm{bdd}(A)$, and for $\alpha > 0$

$$\ell_\alpha^\Sigma(a/A) = \{b \in \mathrm{bdd}(aA) : \mathrm{tp}(b/\bigcup_{\beta<\alpha} \ell_\beta(a/A)) \text{ is almost } \Sigma\text{-internal}\}.$$

We shall write $\ell_\infty^\Sigma(a/A)$ for $\bigcup_\alpha \ell_\alpha^\Sigma(a/A)$, i.e. the set of all hyperimaginaries $b \in \mathrm{bdd}(aA)$ such that $\mathrm{tp}(b/A)$ is Σ-analysable.

Remark 3.23. So $a \in \ell_\alpha^\Sigma(a/A)$ is and only if $\mathrm{tp}(a/A)$ is Σ-analysable in α steps.

Lemma 3.24. *If $\mathrm{tp}(a/A)$ is Σ-analysable in α steps for some ordinal α or $\alpha = \infty$ and $A \mathop{\smile\!\!\!\!\!\!|}\, b$, put $c = \ell_\alpha^\Sigma(b)$. Then $Aa \mathop{\smile\!\!\!\!\!\!|}_c b$.*

Proof: $\mathrm{Cb}(Aa/b)$ is definable over a Morley sequence $(A_i a_i : i < \omega)$ in $\mathrm{lstp}(Aa/b)$. Then $(A_i : i < \omega) \mathop{\smile\!\!\!\!\!\!|}\, b$ and $\mathrm{tp}(a_i/A_i)$ is Σ-analysable in α steps for all $i < \omega$. The union of these analyses level-by-level gives us a Σ-analysis of $\mathrm{Cb}(Aa/b)$ over $(A_i : i < \omega)$ in α steps. As $(A_i : i < \omega) \mathop{\smile\!\!\!\!\!\!|}\, \mathrm{Cb}(Aa/b)$, we obtain that $\mathrm{Cb}(Aa/b)$ is analysable over $\emptyset$ in α steps, and must be contained in c. Thus $Aa \mathop{\smile\!\!\!\!\!\!|}_c b$. $\square$

Corollary 3.25. *If $c = \ell_\infty^\Sigma(b)$, then $\mathrm{tp}(b/c)$ is foreign to all Σ-analysable types.*

Proof: We apply Lemma 3.24 over c for $\alpha = \infty$, noting that $\ell_\infty^\Sigma(b/c) = \ell_\infty^\Sigma(b) = c$. $\square$

The next proposition is well-known for hyperimaginaries in simple theories, even without the restriction on SU_p-rank: If $\mathrm{tp}(a/A)$ is non-orthogonal to a regular type $p \in S(\emptyset)$, then there is a p-internal $a_0 \in \mathrm{bdd}(aA) \setminus \mathrm{cl}_p(A)$: If $B \mathop{\smile\!\!\!\!\!\!|}_A a$ and $b \models p|AB$ with $b \not\mathop{\smile\!\!\!\!\!\!|}_{AB} a$, just take $a_0 = \mathrm{Cb}(bB/aA)$. In fact, if we just require $a_0 \in \mathrm{bdd}(aA) \setminus \mathrm{bdd}(A)$, we do not even need p to be regular [13, Propositon 3.4.14]. However, as for ultraimaginary a the canonical base does not make sense, we have to work harder.

Proposition 3.26. *Let T be simple. Suppose b_E is an ultraimaginary non-orthogonal to some regular type $p \in S(\emptyset)$, and $SU_p(\ell_1^p(b)) < \omega$. Then there is an almost p-internal ultraimaginary $e \in \mathrm{bdd}^u(b_E) \setminus \mathrm{bdd}^u(\mathrm{cl}_p(\emptyset))$. Moreover, $e \in \mathrm{bdd}^u(\ell_1^p(b))$.*

Proof: Let $c = \ell_1^p(b)$. Define an $\emptyset$-invariant relation R on $\mathrm{tp}(c)$ by

$$R(c', c'') \quad \Leftrightarrow \quad \exists b'b'' \; [b'c' \equiv b''c'' \equiv bc \wedge E(b', b'')].$$

This is reflexive and symmetric; moreover for $c' \mathop{\smile\!\!\!\!\!\!\,|}_{c''} c'''$ with $R(c', c'')$ and $R(c'', c''')$ we can find b', b'', b^*, b''' with

$$b'c' \equiv b''c'' \equiv b^*c'' \equiv b'''c''' \equiv bc,$$

such that $E(b', b'')$ and $E(b^*, b''')$ hold. Since c'' is boundedly closed, $b'' \equiv^{lstp}_{c''} b^*$; moreover $b'' \mathop{\smile\!\!\!\!\!\!\,|}_{c''} c'$ and $b^* \mathop{\smile\!\!\!\!\!\!\,|}_{c''} c'''$ by Lemma 3.24. By the Independence Theorem we can assume $b'' = b^*$, so $E(b', b''')$ and $R(c', c''')$ hold. Hence R is generically transitive; let F be its transitive closure. The class c_F is clearly almost p-internal. Moreover, if $E(b', b)$ holds there is c' with $b'c' \equiv bc$. Thus $F(c', c)$ holds, so c_F is bounded over b_E.

Finally, suppose $c_F \in \mathrm{bdd}^u(\mathrm{cl}_p(\emptyset))$. By Proposition 3.18 there is $c' \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{cl}_p(\emptyset)} c$ with $R(c', c)$. Hence there are b', b^* with $b'c' \equiv b^*c \equiv bc$ and $\models E(b', b^*)$. Applying a c-automorphism (and moving c'), we may assume $b = b^*$. Let $A \mathop{\smile\!\!\!\!\!\!\,|} b$ be some parameters and a some realization of p over A with $a \not\mathop{\smile\!\!\!\!\!\!\,|}_A b_E$; we may assume $Aa \mathop{\smile\!\!\!\!\!\!\,|}_b b'$, whence $A \mathop{\smile\!\!\!\!\!\!\,|} bb'$. Moreover $b \mathop{\smile\!\!\!\!\!\!\,|}_c Aa$ by Lemma 3.24, whence $b' \mathop{\smile\!\!\!\!\!\!\,|}_c Aa$. Thus $b' \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{cl}_p(c)} Aa$ by Fact 3.15. Now $c' \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{cl}_p(\emptyset)} c$ yields $c' \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{cl}_p(\emptyset)} \mathrm{cl}_p(c)$, and hence $c' \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{cl}_p(\emptyset)} Aa$. Then $a \mathop{\smile\!\!\!\!\!\!\,|}_A \mathrm{cl}_p(\emptyset)$ implies $a \mathop{\smile\!\!\!\!\!\!\,|}_A c'$. Now $b' \mathop{\smile\!\!\!\!\!\!\,|}_{c'} Aa$ by Lemma 3.24, whence $b' \mathop{\smile\!\!\!\!\!\!\,|}_A a$. As $b_E = b'_E$ we obtain $a \mathop{\smile\!\!\!\!\!\!\,|}_A b_E$, a contradiction. $\square$

Corollary 3.27. *Let e be a supersimple ultraimaginary. Suppose e is non-orthogonal to some regular type p over some set B. Then there is an almost p-internal supersimple $e' \in \mathrm{bdd}^{qfu}(Be) \setminus \mathrm{bdd}^{qfu}(\mathrm{cl}_p(B))$.*

Proof: Let a be a representative of e with $SU(a) < \infty$ and put $b = \mathrm{Cb}(a/B)$. Then $SU(b) < \infty$, as b is bounded over a finite initial segment of a Morley sequence in $\mathrm{lstp}(a/B)$. Now $e \mathop{\smile\!\!\!\!\!\!\,|}_b B$, so $\mathrm{tp}(e/b)$ is non-orthogonal to p. Note that $SU_p(\ell_1^p(a/b)/b)$ is finite by supersimplicity. By Proposition 3.26 applied over b there is an almost p-internal ultraimaginary $e' \in \mathrm{bdd}^u(be) \setminus \mathrm{bdd}^u(\mathrm{cl}_p(b))$; moreover $e' \in \mathrm{bdd}^u(\ell_1^p(a/b)) \subseteq \mathrm{bdd}(ab)$. Thus e' is supersimple, almost p-internal over b and thus over B; it is quasi-finitary by Remark 3.6. $\square$

Proposition 3.28. *Let T be supersimple. If $AB \mathop{\smile\!\!\!\!\!\!\,|} D$ and $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset)$, then $\mathrm{bdd}^{qfu}(AD) \cap \mathrm{bdd}^{qfu}(BD) = \mathrm{bdd}^{qfu}(D)$.*

Proof: We may assume that A, B and D are boundedly closed. Consider

$$e \in (\mathrm{bdd}^{qfu}(AD) \cap \mathrm{bdd}^{qfu}(BD)) \setminus \mathrm{bdd}^{qfu}(D).$$

Let p be a regular type of least SU-rank non-orthogonal to e over D. This exists by transitivity since e is tame. By Corollary 3.27 we may assume that e is almost p-internal of finite SU_p-rank over D; let a' be a representative which is almost p-internal over D. Put $a = \mathrm{Cb}(a'D/A)$. As $a \mathop{\smile\!\!\!\!\!\!\;|\;} D$ we obtain that $\mathrm{tp}(a)$ is almost p-internal; note that $SU(a) < \infty$. Since $e \mathop{\smile\!\!\!\!\!\!\;|\;}_{aD} A$, Lemma 3.4 implies that $e \in \mathrm{bdd}^{qfu}(aD)$. So we may assume that $A = \mathrm{bdd}(a)$ and $SU_p(A) < \omega$. Moreover, we may assume that $D = \mathrm{bdd}(\mathrm{Cb}(aa'/D))$ is the bounded closure of a finite set.

Let $(A_i : i < \omega)$ be a Morley sequence in $\mathrm{lstp}(A/BD)$ with $A_0 = A$, and put $B' = \mathrm{bdd}(A_1A_2)$. Then B' is almost p-internal of finite SU_p-rank. Since $e \in \mathrm{bdd}^{qfu}(AD) \cap \mathrm{bdd}^{qfu}(BD)$ we have $e \in \mathrm{bdd}^{qfu}(A_iD)$ for all $i < \omega$. Let e' be the set of $B'D$-conjugates of e, again a quasi-finitary ultraimaginary. Since any $B'D$-conjugate of e is again in

$$\begin{aligned}\mathrm{bdd}^{qfu}(A_1D) \cap \mathrm{bdd}^{qfu}(A_2D) &= \mathrm{bdd}^{qfu}(BD) \cap \mathrm{bdd}^{qfu}(A_1D)\\ &= \mathrm{bdd}^{qfu}(BD) \cap \mathrm{bdd}^{qfu}(AD),\end{aligned}$$

we have $e' \in \mathrm{dcl}^{qfu}(B'D) \cap \mathrm{bdd}^{qfu}(AD)$. Moreover, $B' \mathop{\smile\!\!\!\!\!\!\;|\;}_{BD} A$, whence $B' \mathop{\smile\!\!\!\!\!\!\;|\;}_{B} A$ and

$$\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B') \subseteq \mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset).$$

Choose $A' \equiv^{lstp}_{AD} B'$ with $A' \mathop{\smile\!\!\!\!\!\!\;|\;}_{AD} B'$. Then

$$e' \in \mathrm{dcl}^{qfu}(A'D) \cap \mathrm{dcl}^{qfu}(B'D).$$

Now, $D \mathop{\smile\!\!\!\!\!\!\;|\;}_{B} A$ implies $D \mathop{\smile\!\!\!\!\!\!\;|\;}_{B} AB'$; as $D \mathop{\smile\!\!\!\!\!\!\;|\;} B$ we get $D \mathop{\smile\!\!\!\!\!\!\;|\;} ABB'$. Hence $D \mathop{\smile\!\!\!\!\!\!\;|\;}_{A} B'$, whence $A' \mathop{\smile\!\!\!\!\!\!\;|\;}_{A} B'$ and

$$\mathrm{bdd}^{qfu}(A') \cap \mathrm{bdd}^{qfu}(B') \subseteq \mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B') = \mathrm{bdd}^{qfu}(\emptyset).$$

We may assume $e' = (A'D)_E$ for some $\emptyset$-invariant equivalence relation E. Define a $\emptyset$-invariant reflexive and symmetric relation R on $\mathrm{lstp}(A')$ by

$$R(X,Y) \Leftrightarrow \exists\, Z\ [XZ \equiv YZ \equiv A'D \wedge Z \mathop{\smile\!\!\!\!\!\!\;|\;} XY \wedge E(XZ,YZ)].$$

By the independence theorem, if $A_1 \mathop{\smile\!\!\!\!\!\!\;|\;}_{A_2} A_3$ such that $R(A_1,A_2)$ and $R(A_2,A_3)$ hold, we have $R(A_1,A_3)$. Hence R is generically transitive; let E' be the transitive closure of R. Clearly $A'_{E'}$ is quasi-finitary.

Next, consider $A'' \equiv_{B'} A'$ with $A'' \mathop{\smile\!\!\!\!\!\!\;|\;}_{B'} A'$. By the independence theorem there is D' with $A'D \equiv_{B'} A'D' \equiv_{B'} A''D'$ and $D' \mathop{\smile\!\!\!\!\!\!\;|\;}_{B'} A'A''$. Then

$D' \perp B'$, whence $D' \perp A'A''$ and $(A'D')_E = (A''D')_E \in \mathrm{dcl}^{qfu}(B'D')$. Therefore $E'(A', A'')$ holds and $A'_{E'} \in \mathrm{dcl}^{qfu}(B')$. Thus

$$A'_{E'} \in \mathrm{dcl}^{qfu}(A') \cap \mathrm{dcl}^{qfu}(B') \subseteq \mathrm{bdd}^{qfu}(\emptyset).$$

By Proposition 3.18 there is $A'' \perp_{\mathrm{cl}_p(\emptyset)} A'$ with $R(A', A'')$. Let D' witness $R(A', A'')$. Then $D' \equiv_{A'} D$, so we may assume $D' = D$. Since $\mathrm{cl}_p(D) \perp_{\mathrm{cl}_p(\emptyset)} \mathrm{cl}_p(A'A'')$ and $\mathrm{cl}_p(A') \perp_{\mathrm{cl}_p(\emptyset)} \mathrm{cl}_p(A'')$ we obtain

$$\mathrm{cl}_p(A') \underset{\mathrm{cl}_p(\emptyset)}{\perp} \mathrm{cl}_p(A'')\mathrm{cl}_p(D)$$

and hence $A' \perp_{\mathrm{cl}_p(D)} A''$. But now

$$e' = (A'D)_E = (A''D)_E \in \mathrm{dcl}^{qfu}(A'D) \cap \mathrm{dcl}^{qfu}(A''D) \subseteq \mathrm{bdd}^{qfu}(\mathrm{cl}_p(D))$$

by Lemma 3.4. Since $e \in \mathrm{bdd}^{qfu}(e')$, this contradicts non-orthogonality of e to p over D. □

Remark 3.29. Again, the proof of the hyperimaginary analogue of Proposition 3.28 for simple theories uses canonical bases and does not generalize.

4. Elimination of ultraimaginaries

One cannot avoid the non-tame ultraimaginaries of Example 2.8 which do not satisfy finite character and hence cannot be eliminated. Similarly, on a type of rank ω we cannot eliminate the relation of having mutually finite rank over each other (Example 2.9), since the rank over a class modulo such a relation is not defined. We thus content ourselves with elimination of supersimple ultraimaginaries in a simple theory (and in particular of quasi-finitary ultraimaginaries in a supersimple theory) up to rank of lower order of magnitude. This seems to be optimal, given the examples cited.

Definition 4.1. Let e be ultraimaginary. We shall say that $SU(a/e) < \omega^\alpha$ if for all representatives b of e we have $SU(a/b) < \omega^\alpha$. Conversely, $SU(e/a) < \omega^\alpha$ if there is a representative b with $SU(b/a) < \omega^\alpha$.

Remark 4.2. This does not mean that we define the value of $SU(a/e)$ or of $SU(e/a)$. In fact, one might define

$$SU(e/a) = \min\{SU(b/a) : \ b \text{ a representative of } e\},$$

but this suggests a precision I am not sure exists.

On the other hand, Example 2.9 shows that

$$SU(a/e) = \sup\{SU(a/b) : \ b \text{ a representative of } e\}$$

is not a good definition, as the rank of a point a over its connected component $e = a_E$ would be ω, i.e. the same as $SU(a)$.

Lemma 4.3. *Let e be ultraimaginary. $SU(e/a) < \omega^0$ if and only if $e \in \mathrm{bdd}^u(a)$, and $SU(a/e) < \omega^0$ if and only if $a \in \mathrm{bdd}(e)$.*

Proof: If b is a representative of e with $SU(b/a) < \omega^0$, then $b \in \mathrm{bdd}(a)$, so $e \in \mathrm{bdd}^u(a)$. If $e \in \mathrm{bdd}^u(a)$, then $e \in \mathrm{dcl}^u(\mathrm{bdd}(a))$, so $b = \mathrm{bdd}(a)$ is a representative of e with $SU(b/a) < \omega^0$.

If $a \notin \mathrm{bdd}(e)$, then there are arbitrarily many e-conjugates of a. Then for any representative b of e there is some e-conjugate a' of a which is not in $\mathrm{bdd}(b)$. Let b' be the image of b under an e-automorphism mapping a' to a. Then b' is a representative of e, and $SU(a/b') \geq \omega^0$. On the other hand, if $a \in \mathrm{bdd}(e)$, then $a \in \mathrm{bdd}(b)$ for any representative b of e, whence $SU(a/b) < \omega^0$. $\square$

Definition 4.4. An ultraimaginary e can be *α-eliminated* if there is a representative a with $SU(a/e) < \omega^\alpha$. A supersimple theory has *feeble elimination of ultraimaginaries* if for all ordinals α, all quasi-finitary ultraimaginaries of rank $< \omega^{\alpha+1}$ can be α-eliminated.

Remark 4.5. 0-elimination is usually called *weak* elimination; in the presence of imaginaries this equals full elimination. I do not know what the definition of feeble elimination of ultraimaginaries should be in general for simple theories — but then their whole theory is much more problematic.

Theorem 4.6. *If e is ultraimaginary with $SU(e) < \omega^{\alpha+1}$, then e can be α-eliminated. A supersimple theory has feeble elimination of ultraimaginaries; a supersimple theory of finite rank has elimination of quasi-finitary ultraimaginaries.*

Proof: Let a be a representative of e of minimal rank. Since $SU(e) < \omega^{\alpha+1}$ we have $SU(a) < \omega^{\alpha+1}$. Suppose $SU(a/e) \geq \omega^\alpha$, so there is some representative b of e with $SU(a/b) \geq \omega^\alpha$. Let P be the family of regular types of SU-rank ω^α. Then $SU_P(a) < \omega$ and $SU_P(a/b) = n > 0$; we choose b such that n is maximal. Consider $a' \equiv_b^{lstp} a$ with $a' \mathop{\smile\!\!\!\!\!\!|}_b a$. Since $e \in \mathrm{dcl}^u(b)$ we have $e \in \mathrm{dcl}^u(a')$. By maximality of n,

$$SU_P(a/a') \leq n = SU_P(a/b) = SU(a/a'b) \leq SU_P(a/a'),$$

so equality holds. By lemma 3.17 we have

$$a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_P(a')} \mathrm{cl}_P(b).$$

On the other hand, $a \mathop{\smile\!\!\!\!\!\!\;|}_b a'$ implies by the anlogue of Fact 3.15 that

$$a \mathop{\perp}_{\mathrm{cl}_P(b)} \mathrm{cl}_P(a'),$$

so

$$c = \mathrm{Cb}(a/\mathrm{cl}_P(b)\mathrm{cl}_P(a')) \subseteq \mathrm{cl}_P(b) \cap \mathrm{cl}_P(a').$$

Then $a \perp_c b$, so $e \in \mathrm{bdd}^u(c)$ by Lemma 3.4. On the other hand, $c \in \mathrm{cl}_P(a') \cap \mathrm{cl}_P(b)$ implies $SU(c/a') < \omega^\alpha$, and $SU(c/b) < \omega^\alpha$. Then $SU(a'/c) \geq SU(a'/cb) \geq \omega^\alpha$ since $SU(a'/b) \geq \omega^\alpha$. It follows that

$$SU(a) = SU(a') \geq SU(c) + \omega^\alpha.$$

In particular $\mathrm{bdd}(c)$ is a representative for e of lower rank, a contradiction. $\square$

Remark 4.7. Let p be a regular type (or type of weight 1). Then two realizations a and b of p are independent if and only if $\mathrm{bdd}^{qfu}(a) \cap \mathrm{bdd}^{qfu}(b) = \mathrm{bdd}^{qfu}(\emptyset)$: One direction is Lemma 3.4, the other follows from the observation that dependence is an invariant equivalence relation on realizations of p. However, this does not hold for all types: By elimination of quasifinite ultraimaginaries, it is in particular false in non one-based theories of finite rank.

5. Decomposition

In this section we shall give ultraimaginary proofs of some of Chatzidakis' results from [6] around the weak canonical base property, and suitable generalisations to the supersimple case. As before, Σ will be an $\emptyset$-invariant family of partial types in a simple theory.

Recall that a and b are domination-equivalent over A, denoted $a \underline{\square}_A b$, if for any c we have $c \perp_A a \Leftrightarrow c \perp_A b$. The following lemma is folklore, but we give a proof for completeness.

Lemma 5.1. (1) *Suppose $a \underline{\square}\, b$. If $c \perp a$ and $c \perp b$, then $a \underline{\square}_c\, b$.*
(2) *Suppose $a \underline{\square}_c\, b$. If $c \perp ab$ then $a \underline{\square}\, b$.*
(3) *Suppose $a \underline{\square}_c\, b$. If $\mathrm{tp}(a)$ and $\mathrm{tp}(b)$ are foreign to $\mathrm{tp}(c)$, then $a \underline{\square}\, b$.*

Proof:

(1) Consider any d with $d \not\perp_c a$. Then $cd \not\perp a$, whence $cd \not\perp b$. Now $b \perp c$ implies $b \not\perp_c d$. The converse follows by symmetry.

(2) Consider any d with $d \not\perp a$. Clearly we may assume $d \perp_{ab} c$, whence $abd \perp c$. Since $a \perp c$ we get $d \not\perp_c a$, whence $d \not\perp_c b$ and $cd \not\perp b$. But $c \perp_d b$, so $d \not\perp b$; the converse follows by symmetry.

(3) Consider any d with $d \not\perp a$. Since $a \perp c$ we get $d \not\perp_c a$, whence $d \not\perp_c b$ and $cd \not\perp b$. If $b \perp d$, then $b \perp_d c$ by foreigness, whence $b \perp cd$, a contradiction. So $b \not\perp d$; the converse follows by symmetry. □

For the following definitions, we require the notion of Σ-closure alluded to in Remark 3.16.

Definition 5.2. For an ordinal α we put

$$\mathrm{cl}_\Sigma^\alpha(A) = \{a : \mathrm{tp}(a/A) \text{ is } \Sigma\text{-analysable in } \alpha \text{ steps}\}.$$

The Σ-*closure* of A is $\mathrm{cl}_\Sigma(A) = \mathrm{cl}_\Sigma^\infty(A) = \bigcup_\alpha \mathrm{cl}_\Sigma^\alpha(A)$.

Remark 5.3. Note that $\ell_\alpha^\Sigma(a/A) = \mathrm{cl}_\Sigma^\alpha(A) \cap \mathrm{bdd}(aA)$. In particular, $a \perp_{\ell_\alpha^\Sigma(a/A)} \mathrm{cl}_\Sigma^\alpha(A)$ by Lemma 3.24.

Proposition 5.4. *Let a and b be domination-equivalent over $\mathrm{cl}_\Sigma^\alpha(\emptyset)$, where a is quasi-finite and $\mathrm{bdd}^{qfu}(a) \cap \mathrm{bdd}^{qfu}(b) = \mathrm{bdd}^{qfu}(\emptyset)$. Then $ab \in \mathrm{cl}_\Sigma^\alpha(\emptyset)$.*

Proof: Note that by Lemma 3.24 the domination-equivalence of a and b over $\mathrm{cl}_\Sigma^\alpha(\emptyset)$ means that for any d and $D = \ell_\alpha^\Sigma(abd)$ we have

$$a \mathop{\perp}_D d \quad \Leftrightarrow \quad b \mathop{\perp}_D d.$$

Clearly, domination-equivalence over $\mathrm{cl}_\Sigma^\alpha(\emptyset)$ is an $\emptyset$-invariant equivalence relation E on $\mathrm{lstp}(a)$. Let $a' \equiv_b^{lstp} a$ with $a' \perp_b a$. Then $E(a', a)$ holds. But

$$\mathrm{bdd}^{qfu}(a) \cap \mathrm{bdd}^{qfu}(a') \subseteq \mathrm{bdd}^{qfu}(a) \cap \mathrm{bdd}^{qfu}(b) = \mathrm{bdd}^{qfu}(\emptyset).$$

Hence $(a)_E = (a')_E \in \mathrm{bdd}^{qfu}(\emptyset)$, and there is $a'' \perp a$ with $E(a, a'')$. By Lemma 3.24 we have $a \perp_{\ell_\alpha^\Sigma(a)} a''\ell_\alpha^\Sigma(aa'')$, whence $a \perp_{\ell_\alpha^\Sigma(aa'')} a''$. By domination-equivalence, $a \perp_{\ell_\alpha^\Sigma(aa'')} a$, that is $a \in \ell_\alpha^\Sigma(aa'')$, whence $a \in \mathrm{cl}_\Sigma^\alpha(\emptyset)$. Similarly, $b \in \mathrm{cl}_\Sigma^\alpha(\emptyset)$. □

Corollary 5.5. *Let A, B, a, b be (hyperimaginary) sets, such that a is quasi-finite, $\mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(Bb) = \mathrm{bdd}^{qfu}(\emptyset)$, and a and b are interbounded over AB. Suppose AB is Σ-analysable in α steps for some ordinal α or $\alpha = \infty$. Then a and b are Σ-analysable in α steps.*

Proof: Since a and b are interbounded over AB, they are domination-equivalent over $\mathrm{cl}_\Sigma^\alpha(\emptyset)$. Now apply Proposition 5.4. □

Remark 5.6. By Theorem 4.6, if $SU(Aa)$ or $SU(Bb)$ is finite, then $\mathrm{bdd}(Aa) \cap \mathrm{bdd}(Bb) = \mathrm{bdd}(\emptyset)$ implies $\mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(Bb) = \mathrm{bdd}^{qfu}(\emptyset)$, and we recover [6, Lemma 1.15 and Lemma 1.22] for $\alpha = \infty$ and $\alpha = 1$.

Fact 5.7. [10, Theorem 3.4(3)] *Let Σ' be an $\emptyset$-invariant subfamily of Σ. Suppose* $\mathrm{tp}(a)$ *is Σ-analysable, but foreign to $\Sigma \setminus \Sigma'$. Then a and $\ell_1^{\Sigma'}(a)$ are domination-equivalent.*

Corollary 5.8. *Let $A \subseteq \mathrm{bdd}(\mathrm{Cb}(B/A))$ consist of quasi-finite hyperimaginaries, with $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset)$. If A is Σ-analysable and Σ' is the subset of one-based partial types in Σ, then A is analysable in $\Sigma \setminus \Sigma'$.*

Proof: Suppose A is not analysable in $\Sigma \setminus \Sigma'$. For every finite tuple $\bar a \in A$ put $c_{\bar a} = \mathrm{Cb}(B/\bar a)$, and let $C = \bigcup\{c_{\bar a} : \bar a \in A\}$. Then $A \perp_C B$, as for any $\bar a \in A$ and C-indiscernible sequence $(B_i : i < \omega)$ in $\mathrm{tp}(B/C)$ the set $\{\pi(\bar x, B_i) : i < \omega\}$ is consistent, where $\pi(\bar x, B) = \mathrm{tp}(\bar a/B)$, since $\pi(\bar x, B)$ does not fork over $c_{\bar a} \subseteq C$. So $A \subseteq \mathrm{bdd}(C)$; as A is not analysable in $\Sigma \setminus \Sigma'$, neither is C, and there is $\bar a \in A$ such that $c = c_{\bar a}$ is not analysable in $\Sigma \setminus \Sigma'$. Clearly $c \subseteq \mathrm{bdd}(\bar a)$ is quasi-finite and $c = \mathrm{Cb}(B/c)$. Replacing A by c we may thus assume that A is quasi-finite.

Let $A' \subseteq \mathrm{bdd}(A)$ and $B' \subseteq \mathrm{bdd}(B)$ be maximally analysable in $\Sigma \setminus \Sigma'$. So $\mathrm{tp}(A/A')$ and $\mathrm{tp}(B/B')$ are foreign to $\Sigma \setminus \Sigma'$, and $A \not\subseteq A'$. Since $A = \mathrm{Cb}(B/A)$ we get $A \not\perp_{A'} B$; as $A \perp_{A'} B'$ by foreignness to $\Sigma \setminus \Sigma'$, we obtain $A \not\perp_{A'B'} B$. In particular $B \not\subseteq B'$.

By Fact 5.7 the first Σ'-levels $a = \ell_1^{\Sigma'}(A/A')$ and $b = \ell_1^{\Sigma'}(B/B')$ are non-trivial, one-based, and

$$a \mathrel{\underline{\square}}_{A'} A \quad \text{and} \quad b \mathrel{\underline{\square}}_{B'} B.$$

Since $\mathrm{tp}(Aa/A')$ is foreign to $\Sigma \setminus \Sigma'$, we have $Aa \perp_{A'} B'$, whence $a \mathrel{\underline{\square}}_{A'B'} A$ by Lemma 5.1(1). Similarly $b \mathrel{\underline{\square}}_{A'B'} B$. But $A \not\perp_{A'B'} B$, and thus $a \not\perp_{A'B'} b$. Let $a_0 = \mathrm{bdd}(A'a) \cap \mathrm{bdd}(A'B'b)$ and $b_0 = \mathrm{bdd}(B'b) \cap \mathrm{bdd}(A'B'a)$. By one-basedness of $\mathrm{tp}(a/A')$ and $\mathrm{tp}(b/B')$,

$$a \mathop{\perp}_{A'a_0} B'b \qquad \text{and} \qquad b \mathop{\perp}_{B'b_0} A'a.$$

Hence

$$A'B'a \mathop{\perp}_{A'B'a_0} b_0 \qquad \text{and} \qquad A'B'b \mathop{\perp}_{A'B'b_0} a_0.$$

It follows that a_0 and b_0 are interbounded over $A'B'$. We can now apply Corollary 5.5 to see that a_0 is analysable in $\Sigma \setminus \Sigma'$, whence $a_0 \in A'$. But then $a \mathop{\smile\!\!\!\!\!\mid}_{A'B'} b$, a contradiction. $\square$

Remark 5.9. In a theory of finite SU-rank, due to weak elimination of quasi-finitary ultraimaginaries, we obtain that for any A, B

$$\mathrm{tp}(\mathrm{Cb}(A/B)/\mathrm{bdd}(A) \cap \mathrm{bdd}(B))$$

is analysable in the collection of non one-based types of SU-rank 1.

Remark 5.10. Without the quasi-finite hypothesis in Proposition 5.4, Corollary 5.5 and Corollary 5.8, the conclusions still hold if we assume that the full ultraimaginary bounded closures intersect trivially.

The following Theorem generalizes [6, Proposition 1.16] to supersimple theories of infinite rank, at the price of demanding that the quasifinite ultraimaginary bounded closures intersect trivially, rather than just the bounded closures. The proof is essentially the same, but we have to work with ultraimaginaries at key steps. Of course, in finite rank this is equivalent, due to elimination of quasifinite hyperimaginaries; moreover, the families Σ_i in the Theorem are just different orthogonality classes of regular types of rank 1.

Definition 5.11. Two $\emptyset$-invariant families Σ and Σ' are *perpendicular* if no realization of a type in Σ can fork with a realisation of a type in Σ'.

Example 5.12. If p and p' are two orthogonal types of SU-rank 1 non-orthogonal to $\emptyset$ (or whose $\emptyset$-conjugates remain orthogonal), then the families of $\emptyset$-conjugates of p and of p' are perpendicular.

Theorem 5.13. *Let T be supersimple. Suppose $A \subseteq \mathrm{bdd}(\mathrm{Cb}(B/A))$ and $B \subseteq \mathrm{bdd}(\mathrm{Cb}(A/B))$, with $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset)$. Let $(\Sigma_i : i \in I)$ be a family of pairwise perpendicular $\emptyset$-invariant families of partial types such that A is analysable in $\bigcup_{i \in I} \Sigma_i$. For $i \in I$ let A_i and B_i be the maximal Σ_i-analysable subset of $\mathrm{bdd}(A)$ and $\mathrm{bdd}(B)$, respectively. Then $A \subseteq \mathrm{bdd}(A_i : i < \alpha)$ and $B \subseteq \mathrm{bdd}(B_i : i < \alpha)$; moreover $A_i = \mathrm{bdd}(\mathrm{Cb}(B_i/A))$ and $B_i = \mathrm{bdd}(\mathrm{Cb}(A_i/B))$. If Σ_i is one-based, then $A_i = B_i = \mathrm{bdd}(\emptyset)$.*

Proof: Since $\mathrm{Cb}(A_i/B)$ is $\mathrm{tp}(A_i)$-analysable and hence Σ_i-analysable, we have $\mathrm{Cb}(A_i/B) \subseteq B_i$; similarly $\mathrm{Cb}(B_i/A) \subseteq A_i$. As the families in $(\Sigma_i : i \in I)$ are perpendicular, we obtain

$$(A_i : i \in I) \mathop{\smile\!\!\!\!\!\mid}_{(B_i : i \in I)} B \quad \text{and} \quad (B_i : i \in I) \mathop{\smile\!\!\!\!\!\mid}_{(A_i : i \in I)} A.$$

Suppose $A \subseteq \mathrm{bdd}(A_i : i \in I)$. Then $B = \mathrm{Cb}(A/B) \subseteq \mathrm{bdd}(B_i : i \in I)$; moreover

$$\begin{aligned}\mathrm{bdd}(A) &= \mathrm{bdd}(\mathrm{Cb}(B/A)) = \mathrm{bdd}(\mathrm{Cb}(B_i/A) : i \in I)\\ &= \mathrm{bdd}(\mathrm{Cb}(B_i/A_i) : i \in I) \subseteq \mathrm{bdd}(A_i : i \in I) = \mathrm{bdd}(A)\end{aligned}$$

again by perpendicularily. Hence $\mathrm{bdd}(\mathrm{Cb}(B_i/A_i)) = A_i$, and similarly $\mathrm{bdd}(\mathrm{Cb}(A_i/B_i)) = B_i$. But if Σ_i is one-based, then

$$B_i = \mathrm{bdd}(\mathrm{Cb}(A_i/B_i)) \subseteq \mathrm{bdd}(A_i) \cap \mathrm{bdd}(B_i) = \mathrm{bdd}(\emptyset)\ ;$$

similarly $A_i = \mathrm{bdd}(\emptyset)$.

Put $\bar{A} = \mathrm{bdd}(A_i : i \in I)$ and $\bar{B} = \mathrm{bdd}(B_i : i \in I)$. It remains to show that $A \subseteq \bar{A}$. So suppose not. As in the proof of Corollary 5.8 put $c_{\bar{a}} = \mathrm{Cb}(B/\bar{a})$ for every finite tuple $\bar{a} \in A$, and let $C = \bigcup\{c_{\bar{a}} : \bar{a} \in A\}$. Then again $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \subseteq \mathrm{bdd}(C)$; moreover $c_{\bar{a}} = \mathrm{Cb}(B/c_{\bar{a}})$. Since A is not contained in $\bar{A}$, neither is C. Hence there is $\bar{a} \in A$ such that $c = c_{\bar{a}} \notin \bar{A}$. As the maximal Σ_i-analysable subset of $\mathrm{bdd}(c)$ is equal to $\mathrm{bdd}(c) \cap A_i$ we may replace A by c and thus assume that A is quasi-finite. Similarly, we may assume that B is quasi-finite.

Since $A = \mathrm{Cb}(B/A) \not\subseteq \bar{A}$, we have $A \not\mathop{\smile\!\!\!\!\!\!|}_{\bar{A}} B$; as $A \mathop{\smile\!\!\!\!\!\!|}_{\bar{A}} \bar{B}$ we obtain $A \not\mathop{\smile\!\!\!\!\!\!|}_{\bar{A}\bar{B}} B$. Let $(b_j : j < \alpha)$ be an analysis of B over $\bar{B}$ such that for every $j < \alpha$ the type $\mathrm{tp}(b_j/\bar{B}, b_\ell : \ell < j)$ is Σ_{i_j}-analysable for some $i_j \in I$. Let k be minimal with $A \not\mathop{\smile\!\!\!\!\!\!|}_{\bar{A}\bar{B}} (b_j : j \leq k)$. Then $A \mathop{\smile\!\!\!\!\!\!|}_{\bar{A}} \bar{B}, (b_j : j < k)$ and $\mathrm{Cb}(\bar{B}, (b_j : j \leq k)/A)$ is almost Σ_{i_k}-internal over $\bar{A}$. Put $A' = \ell_1^{\Sigma_{i_k}}(A/\bar{A})$ and $B' = \ell_1^{\Sigma_{i_k}}(B/\bar{B})$. Then $A' \not\subseteq \bar{A}$, and $\mathrm{Cb}(A'/B) \subseteq B'$ since $\bar{A} \mathop{\smile\!\!\!\!\!\!|}_{\bar{B}} B$. Similarly $\mathrm{Cb}(B'/A) \subseteq A'$. Moreover $A' \not\mathop{\smile\!\!\!\!\!\!|}_{\bar{A}\bar{B}} B$, whence $A' \not\mathop{\smile\!\!\!\!\!\!|}_{\bar{A}\bar{B}} B'$. Replacing A by $\mathrm{Cb}(B'/A) = \mathrm{Cb}(B'/A')$ and B by $\mathrm{Cb}(A'/B) = \mathrm{Cb}(A'/B')$ we may assume that $\mathrm{tp}(A/\bar{A})$ and $\mathrm{tp}(B/\bar{B})$ are both almost Σ_k-internal (where we write k instead of i_k for ease of notation).

Claim. $\mathrm{bdd}^{qfu}(AB_k) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(B_k)$.

Proof of Claim: Suppose not. As B is analysable in $\bigcup_{i\in I}\Sigma_i$, Corollary 3.27 yields some $i \in I$ and

$$d \in (\mathrm{bdd}^{qfu}(AB_k) \cap \mathrm{bdd}^{qfu}(B)) \setminus \mathrm{bdd}^{qfu}(B_k)$$

such that d is almost Σ_i-internal over B_k; since $\mathrm{tp}(B/B_k)$ is foreign to Σ_k we have $i \neq k$. Hence $A \mathop{\smile\!\!\!\!\!\!|}_{\bar{A}B_k} d$, whence $d \in \mathrm{bdd}^{qfu}(\bar{A}B_k)$ by Lemma 3.4. But $\bar{A} = \mathrm{bdd}(A_i : i \in I)$ and $d \mathop{\smile\!\!\!\!\!\!|}_{A_iB_k} \bar{A}$ by almost Σ_i-internality of d over B_k, whence $d \in \mathrm{bdd}^{qfu}(A_iB_k)$. If $B_kd \mathop{\smile\!\!\!\!\!\!|} A_i$, then

$d \mathop{\smile\!\!\!\!\!\!\perp}_{B_k} A_i$ and $d \in \mathrm{bdd}^{qfu} B_k$ by Lemma 3.4, contradicting the choice of d. Therefore $B_k d \not\mathop{\smile\!\!\!\!\!\!\perp} A_i$; by Corollary 3.27 there is almost Σ_i-internal

$$d' \in \mathrm{bdd}^{qfu}(B_k d) \setminus \mathrm{bdd}^{qfu}(\emptyset).$$

Note that $d' \in \mathrm{bdd}^{qfu}(A_i B_k) \cap \mathrm{bdd}^{qfu}(B)$. But then $d' A_i \mathop{\smile\!\!\!\!\!\!\perp} B_k$, whence $d' \mathop{\smile\!\!\!\!\!\!\perp}_{A_i} B_k$ and

$$d' \in \mathrm{bdd}^{qfu}(A_i) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset),$$

a contradiction. □

Claim. *We may assume $B_k = \mathrm{bdd}(\emptyset)$.*

Proof of Claim: Put $A' = \mathrm{Cb}(B/AB_k)$. Then $B_k \subset A' = \mathrm{Cb}(B/A')$, and $\mathrm{bdd}(A')^{qfu} \cap \mathrm{bdd}(B)^{qfu} = \mathrm{bdd}^{qfu}(B_k)$. If $B' = \mathrm{Cb}(A'/B) = \mathrm{Cb}(A'/B')$, then $A' \mathop{\smile\!\!\!\!\!\!\perp}_{B'} B$ and $A \mathop{\smile\!\!\!\!\!\!\perp}_{A'} B$ yield $B \mathop{\smile\!\!\!\!\!\!\perp}_{B'} A$ by transitivity, since $B' \subseteq \mathrm{bdd}(B)$. Thus $B \subset \mathrm{bdd}(B')$. We add B_k to the language; note that $B_k \neq \mathrm{bdd}(\emptyset)$ implies $B \not\mathop{\smile\!\!\!\!\!\!\perp} B_k$, whence $SU(B'/B_k) < SU(B)$. By induction it is thus sufficient to show that A', B' is still a counterexample over B_k.

So suppose not, and let $\mathrm{bdd}(A') = \mathrm{bdd}(A'_i : i \in I)$ and $\mathrm{bdd}(B') = \mathrm{bdd}(B'_i : i \in I)$ be decompositions, where A'_i and B'_i are maximally Σ_i-analysable over B_k in $\mathrm{bdd}(A')$ and $\mathrm{bdd}(B')$, respectively. So B'_k is Σ_k-analysable, whence $B'_k = B_k \subseteq \bar{B}$ by maximality. Since $B \subseteq \mathrm{bdd}(B')$ is almost Σ_k-internal over $\bar{B}$ and $(B'_i : i \neq k)$ is foreign to Σ_k, we get $B \mathop{\smile\!\!\!\!\!\!\perp}_{\bar{B}} (B'_i : i \neq k)$, whence $B \subset \bar{B}$, a contradiction. □

By symmetry, we may also assume $A_k = \mathrm{bdd}(\emptyset)$.

Put $B' = \mathrm{Cb}(B/A\bar{B})$. Then $\bar{B} \subseteq \mathrm{bdd}(B')$, and since B is almost Σ_k-internal over $\bar{B}$, so is B'. If $A' = \mathrm{Cb}(B'/A)$, then $B' \mathop{\smile\!\!\!\!\!\!\perp}_{A'} A$ and $A \mathop{\smile\!\!\!\!\!\!\perp}_{B'} B$ yield $A \mathop{\smile\!\!\!\!\!\!\perp}_{A'} B$, since $A' \subseteq \mathrm{bdd}(A)$. Thus $A \subseteq \mathrm{bdd}(A')$. Put $B'' = \mathrm{Cb}(A/B') = \mathrm{Cb}(A/B'')$. Then

$$B'' \subseteq \mathrm{bdd}(B') \subseteq \mathrm{bdd}(A\bar{B}),$$

and B'' is almost Σ_k-internal over $\bar{B}$. Moreover, $\bar{A} \mathop{\smile\!\!\!\!\!\!\perp}_{\bar{B}} B'$, whence

$$\bar{B} = \mathrm{Cb}(\bar{A}/\bar{B}) = \mathrm{Cb}(\bar{A}/B') \subseteq \mathrm{Cb}(A/B') = B''.$$

Finally, $A \mathop{\smile\!\!\!\!\!\!\perp}_{B''} B'$ implies

$$A \subseteq \mathrm{bdd}(\mathrm{Cb}(B'/A)) \subseteq \mathrm{bdd}(\mathrm{Cb}(B''/A)).$$

Claim. $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B'') = \mathrm{bdd}^{qfu}(\emptyset)$.

Proof of Claim: Suppose not. By Corollary 3.27 there is $i \in I$ and
$$d \in (\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B'')) \setminus \mathrm{bdd}^{qfu}(\emptyset)$$
which is almost Σ_i-internal; since A is foreign to Σ_k we have $i \neq k$. As B'' is almost Σ_k-internal over $\bar{B}$ we have $d \mathop{\smile\!\!\!\!\!\mid}_{\bar{B}} B''$, whence
$$d \in \mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(\bar{B}) \subseteq \mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset),$$
a contradiction. □

Thus A, B'' is another counterexample; by induction on $SU(AB/\bar{A}\bar{B})$ we may assume
$$SU(A/\bar{A}\bar{B}) = SU(AB''/\bar{A}\bar{B}) = SU(AB/\bar{A}\bar{B}).$$
Similarly, there is another counterexample A'', B'' with $\bar{A} \subset A'' \subseteq \mathrm{bdd}(\bar{A}B'')$, whence
$$SU(B''/\bar{A}\bar{B}) = SU(A''B''/\bar{A}\bar{B}) = SU(AB''/\bar{A}\bar{B}) = SU(A/\bar{A}\bar{B}).$$
But if $SU(X/Y) = SU(Z/Y)$ with $X \in \mathrm{bdd}(YZ)$, then $X \sqsupseteq_Y Z$. Thus $B'' \sqsupseteq_{\bar{A}\bar{B}} A$, so B'' and A are domination-equivalent over $\mathrm{cl}_{\bigcup_{i\neq k}\Sigma_i}(\emptyset)$ by perpendicularity and Lemma 5.1(1). So AB'' is analysable in $\bigcup_{i\neq k}\Sigma_i$ by Proposition 5.4. But $\mathrm{tp}(A/\bar{A})$ is almost Σ_k-internal, whence foreign to $\bigcup_{i\neq k}\Sigma_i$, yielding the final contradiction. □

Remark 5.14. If $C \mathop{\smile\!\!\!\!\!\mid} AB$, then $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset)$ implies $\mathrm{bdd}^{qfu}(AC) \cap \mathrm{bdd}^{qfu}(BC) = \mathrm{bdd}^{qfu}(C)$ by Lemma 3.28. Hence Theorem 5.13 applies over C; this can serve to refine the decomposition.

Remark 5.15. In the finite rank context, it is easy to achieve the hypothesis of Theorem 5.13, as it suffices work over $\mathrm{bdd}(A) \cap \mathrm{bdd}(B)$. In general, however, if
$$\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) \supsetneq \mathrm{bdd}^{qfu}(\mathrm{bdd}(A) \cap \mathrm{bdd}(B)),$$
there is no hyperimaginary set C with
$$\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(C),$$
as this equality implies $\mathrm{bdd}(C) = \mathrm{bdd}(A) \cap \mathrm{bdd}(B)$. Thus, we cannot work over $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B)$, which is not eliminable. If $SU(A/\mathrm{bdd}(A) \cap \mathrm{bdd}(B)) < \omega^{\alpha+1}$, feeble elimination nevertheless yields
$$\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) \subset \mathrm{bdd}^{qfu}(\mathrm{cl}_\alpha(A) \cap \mathrm{cl}_\alpha(B)),$$
so we can work over α-closed sets, as is done in [10, Theorem 5.4].

Corollary 5.16. *Let T be supersimple, and Σ_1 and Σ_2 two perpendicular $\emptyset$-invariant families of partial types. Suppose a is quasi-finite,* $\mathrm{tp}(a)$ *is analysable in* $\Sigma_1 \cup \Sigma_2$ *and* $\mathrm{tp}(a/A)$ *is* Σ_1*-analysable, with* $\mathrm{bdd}^{qfu}(a) \cap \mathrm{bdd}^{qfu}(A) = \mathrm{bdd}^{qfu}(\emptyset)$*. Then* $\mathrm{tp}(a)$ *is* Σ_1*-analysable.*

Proof: Clearly we may assume that $A = \mathrm{Cb}(a/A)$. If $a' = \mathrm{Cb}(A/a)$, then A is interbounded with $\mathrm{Cb}(a'/A)$. Moreover, as $\mathrm{tp}(a/a')$ is Σ_1-analysable, $\mathrm{tp}(a')$ is Σ_1-analysable if and only if $\mathrm{tp}(a)$ is. So we may assume in addition that $a = \mathrm{Cb}(A/a)$.

By Theorem 5.13 we have

$$\mathrm{bdd}(a) = \mathrm{bdd}(\ell_\infty^{\Sigma_1}(a), \ell_\infty^{\Sigma_2}(a)).$$

Hence $\mathrm{tp}(\ell_\infty^{\Sigma_2}(a)/A)$ is Σ_1-analysable. By perpendicularity,

$$\ell_\infty^{\Sigma_2}(a) \in \mathrm{bdd}(A) \cap \mathrm{bdd}(a) = \mathrm{bdd}(\emptyset).$$

Hence $a \in \ell_\infty^{\Sigma_1}(a)$ is Σ_1-analysable. □

For $SU(a)$ finite, this specialises to [6, Proposition 1.20]

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